NEUTRINO MASS, MIXING, AND OSCILLATION[a]

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Do neutrinos have nonzero masses? If they do, then these masses are very tiny, and can be sought only in very sensitive experiments. The most sensitive of these search for *neutrino oscillation*, a quantum interference effect which requires neutrino mass and leptonic mixing. In these lectures, we explain what leptonic mixing is, and then develop the physics of neutrino oscillation, including the general formalism and its application to special cases of practical interest. We also see how neutrino oscillation is affected by the passage of the oscillating neutrinos through matter.

1 Introduction

We humans, and the everyday objects around us, are made of nucleons and electrons, so these particles are the ones most familiar to us. However, for every nucleon or electron, the universe as a whole contains around a billion neutrinos. In addition, each person on earth is bombarded by $\sim 10^{14}$ neutrinos, coming from the sun, every second. Clearly, neutrinos are abundant. Thus, it would be very nice to know something about them.

One of the most basic questions we can ask about neutrinos is: Do they have nonzero masses? Until recently, there was no hard evidence that they do. It was known that, at the heaviest, they are very light compared to the quarks and charged leptons. But for a long time, the natural theoretical prejudice has been that the neutrinos are *not* massless. One reason for this prejudice is that in essentially any of the grand unified theories that unify the strong, electromagnetic, and weak interactions, a given neutrino ν belongs to a large multiplet F, together with at least one charged lepton ℓ, one positively-charged quark q^+, and one negatively-charged quark q^-:

$$F = \begin{bmatrix} \nu \\ \ell \\ q^+ \\ q^- \end{bmatrix} \quad . \tag{1}$$

The neutrino ν is related by a symmetry operation to the other members of F, all of which have nonzero masses. Thus, it would be peculiar if ν did not

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have a nonzero mass as well.

If neutrinos do have nonzero masses, we must understand why they are nevertheless so light. Perhaps the most appealing explanation of their lightness is the "see-saw mechanism".[1] To understand how this mechanism works, let us note that, unlike charged particles, neutrinos may be their own antiparticles. If a neutrino is identical to its antiparticle, then it consists of just two mass-degenerate states: one with spin up, and one with spin down. Such a neutrino is referred to as a Majorana neutrino. In contrast, if a neutrino is distinct from its antiparticle, then it plus its antiparticle form a complex consisting of four mass-degenerate states: the spin up and spin down neutrino, plus the spin up and spin down antineutrino. This collection of four states is called a Dirac neutrino. In the see-saw mechanism, a four-state Dirac neutrino $\mathcal{N}^D$ of mass M_D gets split by "Majorana mass terms" into a pair of two-state Majorana neutrinos. One of these Majorana neutrinos, ν^M, has a small mass M_ν and is identified as one of the observed light neutrinos. The other, N^M, has a large mass M_N characteristic of some high mass scale where new physics beyond the range of current particle accelerators, and responsible for neutrino mass, resides. Thus, N^M has not been observed. The character of the breakup of $\mathcal{N}^D$ into ν^M and N^M is such that $M_\nu M_N \cong M_D^2$. It is reasonable to expect that M_D, the mass of the Dirac particle $\mathcal{N}^D$, is of the order of $M_{\ell \text{ or } q}$, the mass of a typical charged lepton ℓ or quark q, since the latter are Dirac particles too. Then $M_\nu M_N \cong M^2_{\ell \text{ or } q}$. With $M_{\ell \text{ or } q}$ a typical charged lepton or quark mass, and M_N very big, this "see-saw relation" explains why M_ν is very tiny. Note that the see-saw mechanism predicts that each light neutrino ν^M is a two-state Majorana neutrino, identical to its antineutrino.

2 Neutrino Oscillation

To find out whether neutrinos really do have nonzero masses, we need an experimental approach which can detect these masses even if they are very small. The most sensitive approach is the search for neutrino oscillation. Neutrino oscillation is a quantum interference phenomenon in which small splittings between the masses of different neutrinos can lead to large, measurable phase differences between interfering quantum-mechanical amplitudes.

To explain the physics of neutrino oscillation, we must first discuss leptonic "flavor". Suppose a neutrino ν is born in the W-boson decay

$$W^+ \rightarrow \ell_\alpha^+ + \nu \; . \tag{2}$$

Here, $\alpha = e, \mu$ or τ, and ℓ_α^+ is one of the positively charged leptons: $\ell_e^+ \equiv e^+$, $\ell_\mu^+ \equiv \mu^+$, and $\ell_\tau^+ \equiv \tau^+$. Suppose that, without having time to change its

character, the neutrino ν interacts in a detector immediately after its birth in the decay (2), and produces a new charged lepton ℓ_β^- via the reaction ν + *target* $\rightarrow \ell_\beta^-$ + *recoils*. It is found that the "flavor" β of this new charged lepton is always the same as the flavor α of the charged lepton with which ν was born. It follows that the neutrinos produced by the W decays (2) to charged leptons of different flavors must be different objects. We take this fact into account by writing these decays more accurately as

$$W^+ \rightarrow \ell_\alpha^+ + \nu_\alpha \; ; \; \alpha = e, \mu, \tau \; . \tag{3}$$

The neutrino ν_α, called the neutrino of flavor α, is by definition the neutrino produced in leptonic W decay in association with the charged lepton of flavor α. As we have said, when ν_α interacts to create a charged lepton, the latter lepton is always ℓ_α. In neutrino oscillation, a neutrino born in association with a charged lepton ℓ_α of flavor α then travels for some time during which it can alter its character. Finally, it interacts to produce a second charged lepton ℓ_β with a flavor β *different* from the flavor α of the charged lepton with which the neutrino was born. For example, suppose a neutrino is born with a muon in the pion decay $\pi^+ \rightarrow$ Virtual $W^+ \rightarrow \mu^+ + \nu_\mu$. Suppose further that after traveling down a neutrino beamline, this same neutrino interacts in a detector and produces, not another muon, but a τ^-. At birth, the neutrino was a ν_μ. But by the time it interacted in the detector, it had turned into a ν_τ. One describes this metamorphosis by saying the neutrino oscillated from a ν_μ into a ν_τ. As we will see, the probability for it to change its flavor does indeed oscillate with the distance it travels before interacting. As we will also see, the oscillation in vacuum of a neutrino between different flavors requires neutrino mass.

To see how neutrino mass can lead to neutrino oscillation, let us briefly recall the weak interactions of quarks. As we all know, there are three quarks—the u (up), c (charm), and t (top) quarks—which carry a positive electric charge $Q = +2/3$. In addition, there are three quarks—the d (down), s (strange), and b (bottom) quarks—which carry a negative electric charge $Q = -1/3$. Each of these six quarks is a particle of definite mass. As we know, the quarks are arranged into three families or generations, each of which contains one positive quark and one negative quark:

$$\begin{array}{cccc} \mathit{Family}: & 1 & 2 & 3 \\ \mathit{Quarks}: & \begin{pmatrix} u \\ d \end{pmatrix} & \begin{pmatrix} c \\ s \end{pmatrix} & \begin{pmatrix} t \\ b \end{pmatrix} \end{array}$$

However, we know experimentally that under the weak interaction, any of the negative quarks, d, s, or b, can absorb a positively-charged W boson and turn

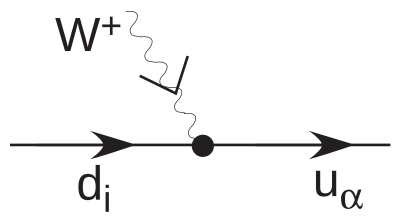

Figure 1: Absorption of a W boson by a quark.

into any of the positive quarks, u, c, or t. This is illustrated in Fig. (1). There, $d_i(i = d, s, b)$ is one of the down-type (negative) quarks. That is, $d_d \equiv d$ is the down quark, $d_s \equiv s$ is the strange quark, and so on. Similarly, $u_\alpha(\alpha = u, c, t)$ is one of the up-type (positive) quarks, with $u_c \equiv c$ being the charm quark, and so on.

In the S(tandard) M(odel) of the electroweak interactions,[2] the W-quark couplings depicted in Fig. (1) are described by the Lagrangian density

$$\mathcal{L}_{udW} = -\frac{g}{\sqrt{2}} \sum_{\substack{\alpha=u,c,t \\ i=d,s,b}} \overline{u_{L\alpha}} \gamma^\lambda V_{\alpha i} d_{Li} W^+_\lambda + \text{h.c.} \quad . \tag{4}$$

Here, the subscript L denotes left-handed chiral projection. For instance, $d_{Ls} \equiv \frac{1}{2}(1 - \gamma_5) d_s$ is the left-handed strange-quark field. The constant g is the semiweak coupling constant, and V is a 3×3 matrix known as the quark mixing matrix. In the SM, V is unitary, because it is basically the matrix for the transformation from one basis of quantum states to another.

The SM interaction (4) is very well confirmed experimentally. With this established behavior of quarks in mind, let us now return to the leptons. Like the quarks, the charged leptons e, μ, and τ are particles of definite mass. However, if leptons behave as quarks do, then the neutrinos ν_e, ν_μ, and ν_τ of definite flavor are *not* particles of definite mass. Let us call the neutrinos which do have definite masses $\nu_i,\ i = 1, 2, \ldots, N$. As far as we know, the number of $\nu_i,\ N$, may exceed the number of charged leptons, three. Now, as we recall, the neutrino ν_α of definite flavor α is the neutrino state that accompanies the definite-mass charged lepton ℓ_α in the decay $W \to \ell_\alpha + \nu_\alpha$. If leptons behave as quarks do, this neutrino state must be a superposition of the neutrinos ν_i of definite mass. To see this, we first note that just as any negative quark d_i of definite mass can absorb a W^+ and turn into any positive quark u_α of definite mass, so it must be possible for any neutrino ν_i of definite mass to absorb a

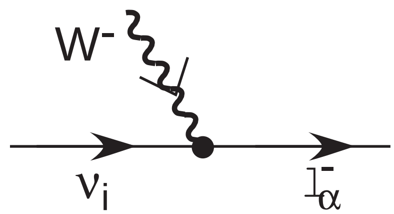

Figure 2: Absorption of a W boson by a neutrino.

W^- and turn into any charged lepton ℓ_α^- of definite mass. This absorption is illustrated in Fig. (2). We expect that in analogy with Eq. (4) for the W-quark couplings, the SM interaction that describes the W-lepton couplings is

$$\mathcal{L}_{\ell\nu W} = -\frac{g}{\sqrt{2}} \sum_{\substack{\alpha=e,\mu,\tau \\ i=1,\ldots,N}} \overline{\ell_{L\alpha}} \gamma^\lambda U_{\alpha i} \nu_{Li} W_\lambda^- - \frac{g}{\sqrt{2}} \sum_{\substack{\alpha \\ i}} \overline{\nu_{Li}} \gamma^\lambda U_{i\alpha}^\dagger \ell_{L\alpha} W_\lambda^+ \ . \quad (5)$$

Here, U is an $N \times N$ unitary matrix which is the leptonic analogue of the quark mixing matrix V. The matrix U is referred to as the leptonic mixing matrix.[3] If $N > 3$, then only the top 3 rows of U enter in the W-lepton interaction, Eq. (5).

The leptonic decays of the W^+ are governed by the second term of $\mathcal{L}_{\ell\nu W}$, Eq. (5). From this term, we see that when $W^+ \to \ell_\alpha^+ +$ "ν_α", the neutrino state $|\nu_\alpha\rangle$ produced in association with the specific definite-mass charged lepton ℓ_α^+ is

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i}^* \, |\nu_i\rangle \ . \quad (6)$$

That is, the "flavor-α" neutrino $|\nu_\alpha\rangle$ produced together with ℓ_α^+ is a coherent superposition of the mass-eigenstate neutrinos $|\nu_i\rangle$, with coefficients which are elements of the leptonic mixing matrix.

What if N is bigger than three? Suppose, for example, that $N = 4$. Then, with the elements of the bottom row of U, $U_{\mathrm{lastrow},i}$, we can construct a neutrino state

$$|\nu_s\rangle \equiv \sum_i U_{\mathrm{lastrow},i}^* \, |\nu_i\rangle \quad (7)$$

which does not couple to any of the 3 charged leptons. This state is called a "sterile" neutrino, which just means that it does not participate in the SM weak interactions. It may, however, participate in other interactions beyond the SM whose effects at present-day energies are too feeble to have been observed.

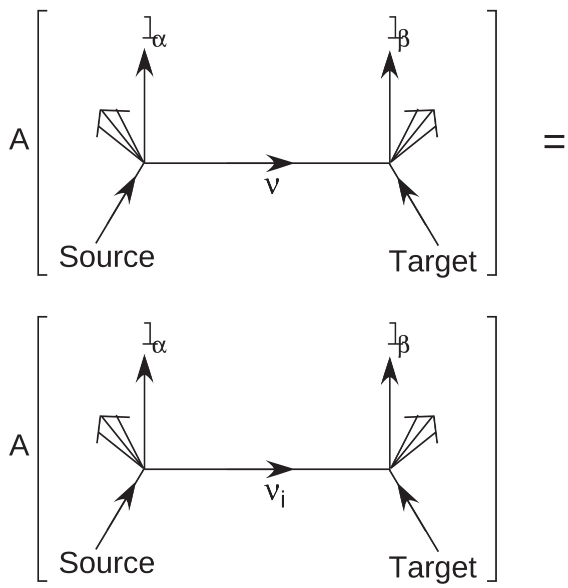

Figure 3: Creation of a neutrino with a charged lepton of flavor α followed by the interaction of this neutrino to produce a charged lepton of flavor β. The "Source" is the particle whose decay creates the neutrino, ℓ_α, and other, unlabelled, particles. The "Target" is the particle struck by the neutrino to produce ℓ_β and other, unlabelled, particles. "A" denotes an amplitude.

Owing to the leptonic mixing described by Eq. (5), when the charged lepton of flavor α is created, the accompanying neutrino can be any of the ν_i. Furthermore, if this ν_i later interacts with some target, it can produce a charged lepton ℓ_β of any flavor β. In such a sequence of events, the neutrino itself is an unseen intermediate state. Thus, as shown in Fig. (3), the amplitude for a neutrino to be born with charged lepton ℓ_α, and then to interact and produce charged lepton ℓ_β, is a coherent sum over the contributions of all the unseen mass eigenstates ν_i.

The birth of a neutrino with charged lepton $\ell_\alpha^{(+)}$ and its subsequent interaction to produce charged lepton $\ell_\beta^{(-)}$ is usually described as the oscillation $\nu_\alpha \to \nu_\beta$ of a neutrino of flavor α into one of flavor β (see earlier discussion).

Using "A" to denote an amplitude, we see from Fig. (3) that

$$A(\nu_\alpha \to \nu_\beta) = \sum_i [A(\text{neutrino born with } \ell_\alpha^+ \text{ is } \nu_i) \times \\ A(\nu_i \text{ propagates }) \, A(\text{when } \nu_i \text{ interacts it makes } \ell_\beta^-)] \ . \quad (8)$$

From $\mathcal{L}_{\ell\nu W}$, Eq. (5), we find that apart from irrelevant factors,

$$A(\text{neutrino born with } \ell_\alpha^+ \text{ is } \nu_i) = U_{\alpha i}^* \ . \quad (9)$$

Similarly,

$$A(\text{when } \nu_i \text{ interacts it makes } \ell_\beta^-) = U_{\beta i} \ . \quad (10)$$

To find the amplitude A(ν_i propagates), we note that in the rest frame of ν_i, where the proper time is τ_i,[4], Schrödinger's equation states that

$$i \frac{\partial}{\partial \tau_i} |\nu_i(\tau_i)\rangle = M_i \, |\nu_i(\tau_i)\rangle \ . \quad (11)$$

Here, M_i is the mass of ν_i. From Eq. (11),

$$|\nu_i(\tau_i)\rangle = e^{-iM_i\tau_i} \, |\nu_i(0)\rangle \ . \quad (12)$$

Now, for propagation over a proper time interval τ_i, A(ν_i propagates) is just the amplitude for finding the original state $|\nu_i(0)\rangle$ in the time-evolved $|\nu_i(\tau_i)\rangle$. That is,

$$A(\nu_i \text{ propagates}) = \langle \nu_i(0) | \nu_i(\tau_i) \rangle = e^{-iM_i\tau_i} \ . \quad (13)$$

In terms of the time t and position L in the laboratory frame, the Lorentz-invariant phase factor $\exp(-iM_i\tau_i)$ is

$$e^{-i(E_i t - p_i L)} \ . \quad (14)$$

Here, E_i and p_i are, respectively, the energy and momentum of ν_i in the laboratory frame. In practice, our neutrino will be highly relativistic, so if it was born at $(t, L) = (0, 0)$, we will be interested in evaluating the phase factor (14) where $t \approx L$, where it becomes

$$e^{-i(E_i - p_i)L} \ . \quad (15)$$

Suppose that the neutrino created with ℓ_α is produced with a definite momentum p, regardless of which ν_i it happens to be. Then, if it is the particular mass eigenstate ν_i, it has total energy

$$E_i = \sqrt{p^2 + M_i^2} \approx p + \frac{M_i^2}{2p} \ , \quad (16)$$

assuming that all the masses M_i are much smaller than p. From (15), we then find that

$$A(\nu_i \text{ propagates}) \approx e^{-i\frac{M_i^2}{2p}L} \ . \tag{17}$$

Alternatively, suppose that our neutrino is produced with a definite energy E, regardless of which ν_i it happens to be.[5] Then, if it is the particular mass eigenstate ν_i, it has momentum

$$p_i = \sqrt{E^2 - M_i^2} \approx E - \frac{M_i^2}{2E} \ . \tag{18}$$

From (15), we then find that

$$A(\nu_i \text{ propagates}) \approx e^{-i\frac{M_i^2}{2E}L} \ . \tag{19}$$

Since highly relativistic neutrinos have $E \approx p$, the propagation amplitudes given by Eqs. (17) and (19) are approximately equal. Thus, it doesn't matter whether our neutrino is created with definite momentum or definite energy.

Collecting the various factors that appear in Eq. (8), we conclude that the amplitude $A(\nu_\alpha \to \nu_\beta)$ for a neutrino of energy E to oscillate from a ν_α to a ν_β while traveling a distance L is given by

$$A(\nu_\alpha \to \nu_\beta) = \sum_i U^*_{\alpha i} e^{-iM_i^2\frac{L}{2E}} U_{\beta i} \ . \tag{20}$$

The probability $P(\nu_\alpha \to \nu_\beta)$ for this oscillation is then given by

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= |A(\nu_\alpha \to \nu_\beta)|^2 \\ &= \delta_{\alpha\beta} - 4\sum_{i>j} \Re(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}) \sin^2(\delta M^2_{ij}\frac{L}{4E}) \\ &\quad +2\sum_{i>j} \Im(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}) \sin(\delta M^2_{ij}\frac{L}{2E}) \ . \end{aligned} \tag{21}$$

Here, $\delta M^2_{ij} \equiv M_i^2 - M_j^2$, and in calculating $|A(\nu_\alpha \to \nu_\beta)|^2$, we have used the unitarity constraint

$$\sum_i U^*_{\alpha i} U_{\beta i} = \delta_{\alpha\beta} \ . \tag{22}$$

Some general comments are in order:

1. From Eq. (21) for $P(\nu_\alpha \to \nu_\beta)$, we see that if all neutrino masses vanish, then $P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta}$, and there is no oscillation from one flavor to another. Neutrino flavor oscillation requires neutrino mass.

2. The probability $P(\nu_\alpha \to \nu_\beta)$ oscillates as a function of L/E. This is why the phenomenon we are discussing is called "neutrino oscillation".

3. From Eq. (20) for $A(\nu_\alpha \to \nu_\beta)$, we see that the L/E dependence of neutrino oscillation arises from interferences between the contributions of the different mass eigenstates ν_i. We also see that the phase of the ν_i contribution is proportional to M_i^2. Thus, the interferences can give us information on neutrino masses. However, since these interferences can only reveal the *relative* phases of the interfering amplitudes, experiments on neutrino oscillation can only determine the splittings $\delta M_{ij}^2 \equiv M_i^2 - M_j^2$, and not the underlying individual neutrino masses. This fact is made perfectly clear by Eq. (21) for $P(\nu_\alpha \to \nu_\beta)$.

4. With the so-far omitted factors of $\hbar$ and c inserted,

$$\delta M_{ij}^2 \frac{L}{4E} = 1.27\, \delta M_{ij}^2(\mathrm{eV}^2) \frac{L(\mathrm{km})}{E(\mathrm{GeV})} \; . \tag{23}$$

   Thus, from Eq. (21) for the oscillation probability $P(\nu_\alpha \to \nu_\beta)$, we see that an oscillation experiment characterized by a given value of L(km) / E(GeV) is sensitive to mass splittings obeying

$$\delta M_{ij}^2(\mathrm{eV}^2) \gtrsim \left[\frac{L(\mathrm{km})}{E(\mathrm{GeV})} \right]^{-1} \; . \tag{24}$$

   To be sensitive to tiny δM_{ij}^2, an experiment must have large L/E. In Table 1, we indicate the δM^2 reach implied by Eq. (24) for experiments working with neutrinos produced in various ways.

5. There are basically two kinds of oscillation experiments: *appearance* experiments, and *disappearance* experiments.

   In an appearance experiment, one looks for the *appearance* in the neutrino beam of neutrinos bearing a flavor not present in the beam initially. For example, imagine that a beam of neutrinos is produced by the decays of charged pions. Such a beam consists almost entirely of muon neutrinos and contains no tau neutrinos. One can then look for the appearance of tau neutrinos, made by oscillation of the muon neutrinos, in this beam.

   In a disappearance experiment, one looks for the *disappearance* of some fraction of the neutrinos bearing a flavor which *is* present in the beam initially. For example, imagine again that a beam of neutrinos is produced by the decays of charged pions, so that almost all the neutrinos

Table 1: The approximate reach in δM^2 of experiments studying various types of neutrinos. Often, an experiment covers a range in L and a range in E. To construct the table, we have used typical values of these quantities.

Neutrinos (Baseline)	L(km)	E(GeV)	$\frac{L(\text{km})}{E(\text{GeV})}$	$\delta M_{ij}^2(\text{eV}^2)$ Reach
Accelerator (Short Baseline)	1	1	1	1
Reactor (Medium Baseline)	1	10^{-3}	10^3	10^{-3}
Accelerator (Long Baseline)	10^3	10	10^2	10^{-2}
Atmospheric	10^4	1	10^4	10^{-4}
Solar	10^8	10^{-3}	10^{11}	10^{-11}

in the beam are muon neutrinos, ν_μ. If one knows the ν_μ flux that is produced initially, one can look to see whether some of this initial ν_μ flux disappears after the beam has traveled some distance, and the muon neutrinos have had a chance to oscillate into other flavors.

6. Even though neutrinos can change flavor through oscillation, the total flux of neutrinos in a beam will be conserved so long as U is unitary. To see this, note that

$$\begin{aligned}\sum_\beta P(\nu_\alpha \to \nu_\beta) &= \sum_\beta |A(\nu_\alpha \to \nu_\beta)|^2 \\ &= \sum_\beta (\sum_i U_{\alpha i}^* U_{\beta i} e^{-iM_i^2 \frac{L}{2E}})(\sum_j U_{\alpha j} U_{\beta j}^* e^{+iM_j^2 \frac{L}{2E}}) \\ &= \sum_{i,j} U_{\alpha i}^* U_{\alpha j} e^{-i\delta M_{ij}^2 \frac{L}{2E}} \sum_\beta U_{\beta i} U_{\beta j}^* \\ &= \sum_i |U_{\alpha i}|^2 = 1 \ . \end{aligned} \tag{25}$$

Here, we have used Eq. (20) for the amplitude $A(\nu_\alpha \to \nu_\beta)$ and the unitarity relations $\sum_\beta U_{\beta i} U_{\beta j}^* = \delta_{ij}$ and $\sum_i |U_{\alpha i}|^2 = 1$. The result $\sum_\beta P(\nu_\alpha \to \nu_\beta) = 1$ means that if one starts with a certain number n of neutrinos of flavor α, then after oscillation the number of neutrinos

that have oscillated away into new flavors $\beta \neq \alpha$, plus the number that have retained the original flavor α, is still n. Note, however, that some of the new flavors that get populated by the oscillation might be sterile. If they are indeed sterile, then the number of "active" neutrinos (i.e., neutrinos that participate in the SM weak interactions) remaining after oscillation will be less than n.

2.1 Special Cases

Let us now apply the general formalism for neutrino oscillation in vacuum to several special cases of practical interest.

The simplest special case of all is two-neutrino oscillation. This occurs when the SM weak interaction, Eq. (5), couples two charged leptons (say, e and μ) to just two neutrinos of definite mass, ν_1 and ν_2, and only negligibly to any other neutrinos of definite mass. It is then easily shown that the 2×2 submatrix

$$\hat{U} = \begin{bmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{bmatrix} \tag{26}$$

of the mixing matrix U must be unitary all by itself. This means that the definite-flavor neutrinos ν_e and ν_μ are composed exclusively of the mass eigenstates ν_1 and ν_2, and do not mix with neutrinos of any other flavor.

From Eq. (20) for the oscillation amplitude, we have for this two-neutrino case

$$e^{+iM_1^2 \frac{L}{2E}} A(\nu_e \to \nu_\mu) = U_{e1}^* U_{\mu 1} + U_{e2}^* U_{\mu 2} e^{-i\delta M_{21}^2 \frac{L}{2E}} \ . \tag{27}$$

Since $\hat{U}$ is unitary, $U_{e1}^* U_{\mu 1} + U_{e2}^* U_{\mu 2} = 0$, so Eq. (27) may be rewritten as

$$\begin{aligned} e^{+iM_1^2 \frac{L}{2E}} A(\nu_e \to \nu_\mu) &= -U_{e2}^* U_{\mu 2} (1 - e^{-i\delta M_{21}^2 \frac{L}{2E}}) \\ &= -2i\, e^{-i\delta M_{21}^2 \frac{L}{4E}} U_{e2}^* U_{\mu 2} \sin(\delta M_{21}^2 \frac{L}{4E}) \ . \end{aligned} \tag{28}$$

Squaring this result, using Eq. (23) to take account of the requisite factors of $\hbar$ and c, we find that the probability $P(\nu_e \to \nu_\mu)$ for a ν_e to oscillate into a ν_μ is given by

$$P(\nu_e \to \nu_\mu) = 4|U_{e2}|^2 |U_{\mu 2}|^2 \sin^2(1.27\, \delta M^2(\mathrm{eV}^2) \frac{L(\mathrm{km})}{E(\mathrm{GeV})}) \ . \tag{29}$$

Here, we have introduced the abbreviation $\delta M_{21}^2 \equiv \delta M^2$. From the $e \leftrightarrow \mu$ symmetry of the right-hand side of Eq. (29), it is obvious that

$$P(\nu_\mu \to \nu_e) = P(\nu_e \to \nu_\mu) \ . \tag{30}$$

The probability $P(\nu_\alpha \to \nu_\alpha)$ that a neutrino ν_α of flavor $\alpha = e$ or μ retains its original flavor is given by

$$\begin{aligned} P(\nu_\alpha \to \nu_\alpha) &= 1 - P(\nu_\alpha \to \nu_{\beta\neq\alpha}) \\ &= 1 - 4|U_{\alpha 2}|^2(1 - |U_{\alpha 2}|^2)\sin^2(1.27\,\delta M^2(\mathrm{eV}^2)\frac{L(\mathrm{km})}{E(\mathrm{GeV})})\ . \end{aligned} \tag{31}$$

To obtain this expression, we have used the conservation of probability, Eq. (25), the "off-diagonal" oscillation probability, Eqs. (29) and (30), and the unitarity relation $|U_{e2}|^2 + |U_{\mu 2}|^2 = 1$.

The unitarity of $\hat{U}$, Eq. (26), implies that it can be written in the form

$$\hat{U} = \begin{bmatrix} e^{i\varphi_1}\cos\theta & e^{i\varphi_2}\sin\theta \\ -e^{i(\varphi_1+\varphi_3)}\sin\theta & e^{i(\varphi_2+\varphi_3)}\cos\theta \end{bmatrix}\ . \tag{32}$$

Here, θ is an angle referred to as the leptonic mixing angle and $\varphi_{1,2,3}$ are phases. From Eq. (32), $4|U_{e2}|^2|U_{\mu 2}|^2 = \sin^2 2\theta$, so that $P(\nu_e \to \nu_\mu)$, Eq. (29), takes the form

$$P(\nu_e \to \nu_\mu) = P(\nu_\mu \to \nu_e) = \sin^2 2\theta \sin^2(1.27\,\delta M^2(\mathrm{eV}^2)\frac{L(\mathrm{km})}{E(\mathrm{GeV})})\ . \tag{33}$$

This is the most-commonly quoted form of the two-neutrino oscillation probability.

A second special case which may prove to be very relevant to the real world is a three-neutrino scenario in which two of the neutrino mass eigenstates are nearly degenerate. That is, the neutrino (Mass)^2 spectrum is as in Fig. (4), where

$$|\delta M^2_{21}| \equiv \delta M^2_{\mathrm{Small}} \ll |\delta M^2_{31}| \cong |\delta M^2_{32}| \equiv \delta M^2_{\mathrm{Big}}\ . \tag{34}$$

All three of the charged leptons, e, μ, and τ, are coupled by the SM weak interaction, Eq. (5), to the neutrinos $\nu_{1,2,3}$.

Suppose that an oscillation experiment has L/E such that $\delta M^2_{\mathrm{Big}} L\,/E$ is of order unity, which implies that $\delta M^2_{\mathrm{Small}} L/E \ll 1$. For this experiment, and $\beta \neq \alpha$, the oscillation amplitude of Eq. (20) is given approximately by

$$e^{iM_3^2\frac{L}{2E}} A(\nu_\alpha \to \nu_{\beta\neq\alpha}) \cong (U^*_{\alpha 1}U_{\beta 1} + U^*_{\alpha 2}U_{\beta 2})e^{i\delta M^2_{32}\frac{L}{2E}} + U^*_{\alpha 3}U_{\beta 3}\ . \tag{35}$$

Using the unitarity constraint of Eq. (22), this becomes

$$\begin{aligned} e^{iM_3^2\frac{L}{2E}} A(\nu_\alpha \to \nu_{\beta\neq\alpha}) &\cong U^*_{\alpha 3}U_{\beta 3}(1 - e^{i\delta M^2_{32}\frac{L}{2E}}) \\ &= -2ie^{i\delta M^2_{32}\frac{L}{4E}} U^*_{\alpha 3}U_{\beta 3}\sin(\delta M^2_{32}\frac{L}{4E})\ . \end{aligned} \tag{36}$$

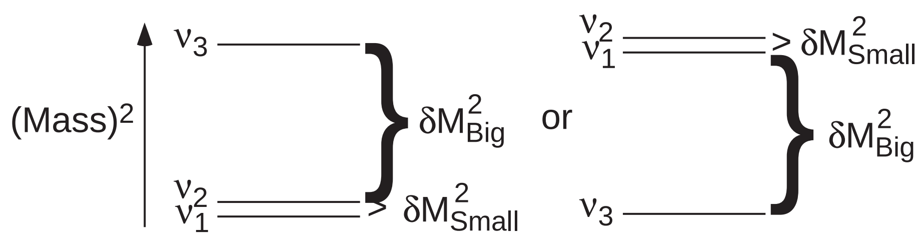

Figure 4: A three neutrino $(\mathrm{Mass})^2$ spectrum in which the $\nu_2-\nu_1$ splitting $\delta M^2_{\mathrm{Small}}$ is much smaller than the splitting $\delta M^2_{\mathrm{Big}}$ between ν_3 and the $\nu_2-\nu_1$ pair. The latter pair may be at either the bottom or the top of the spectrum.

Taking the absolute square of this relation, using $|\delta M^2_{32}| \equiv \delta M^2_{\mathrm{Big}}$, and inserting the omitted factors of $\hbar$ and c, we find that the $\nu_\alpha \to \nu_{\beta\neq\alpha}$ oscillation probability is given by

$$P(\nu_\alpha \to \nu_{\beta\neq\alpha}) = 4|U_{\alpha 3}|^2|U_{\beta 3}|^2 \sin^2(1.27\, \delta M^2_{\mathrm{Big}}(\mathrm{eV}^2)\frac{L(\mathrm{km})}{E(\mathrm{GeV})}) \;. \tag{37}$$

To find the corresponding probability $P(\nu_\alpha \to \nu_\alpha)$ that a neutrino of flavor α retains its original flavor, we simply use the conservation of probability, Eq. (25):

$$P(\nu_\alpha \to \nu_\alpha) = 1 - \sum_{\beta\neq\alpha} P(\nu_\alpha \to \nu_\beta) \;. \tag{38}$$

From Eq. (37) and the unitarity relation $\sum_{\beta\neq\alpha}|U_{\beta 3}|^2 = 1 - |U_{\alpha 3}|^2$, we then find that

$$P(\nu_\alpha \to \nu_\alpha) = 1 - 4|U_{\alpha 3}|^2(1-|U_{\alpha 3}|^2)\sin^2(1.27\, \delta M^2_{\mathrm{Big}}(\mathrm{eV}^2)\frac{L(\mathrm{km})}{E(\mathrm{GeV})}) \;. \tag{39}$$

Comparing Eqs. (37) and (29), and Eqs. (39) and (31), we see that in the three-neutrino scenario with $\delta M^2_{\mathrm{Small}} L/E \ll 1$, the oscillation probabilities are the same as in the two-neutrino case, except that with three neutrinos, the isolated one, ν_3, plays the role played by ν_2 when there are only two neutrinos. This strong similarity between the oscillation probabilities in the two cases is easy to understand. When the three-neutrino case is studied by an experiment for which $\delta M^2_{\mathrm{Small}} L/E \ll 1$, the experiment cannot see the splitting between ν_1 and ν_2. In such an experiment, there appear to be only two neutrinos with distinct masses: the ν_1 and ν_2 pair, which looks like a single neutrino, and ν_3. As a result, the two neutrino oscillation probabilities hold.[6]

3 Neutrino Oscillation in Matter

So far, we have been talking about neutrino oscillation *in vacuum*. However, some very important oscillation experiments are concerned with neutrinos that travel through a lot of matter before reaching the detector. These neutrinos include those made by nuclear reactions in the core of the sun, which traverse a lot of solar material on their way out of the sun towards solar neutrino detectors here on earth. They also include the neutrinos made in the earth's atmosphere by cosmic rays. These atmospheric neutrinos can be produced in the atmosphere on one side of the earth, and then travel through the whole earth before being detected in a detector on the other side. To deal with the solar and atmospheric neutrinos, we need to understand how passage through matter affects neutrino oscillation.

To be sure, the interaction between neutrinos and matter is extremely feeble. Nevertheless, the coherent forward scattering of neutrinos from many particles in a material medium can build up a big effect on the oscillation amplitude.

For both the solar and atmospheric neutrinos, it is a good approximation to take just two neutrinos into account. Furthermore, it is convenient to treat the propagation of neutrinos in matter in terms of an effective Hamiltonian. To set the stage for this treatment, let us first derive the Hamiltonian for travel through vacuum. For the sake of illustration, let us suppose that the two neutrino flavors that need to be considered are ν_e and ν_μ. The most general time-dependent neutrino state vector $|\nu(t)\rangle$ can then be written as

$$|\nu(t)\rangle = \sum_{\alpha=e,\mu} f_\alpha(t)|\nu_\alpha\rangle \ , \tag{40}$$

where $f_\alpha(t)$ is the time-dependent amplitude for the neutrino to have flavor α. If $\mathcal{H}_V$ (short for $\mathcal{H}_{\mathrm{Vacuum}}$) is the Hamiltonian for this two-neutrino system in vacuum, then Schrödinger's equation for $|\nu(t)\rangle$ reads

$$\begin{aligned} i\frac{\partial}{\partial t}|\nu(t)\rangle &= \sum_\alpha i\dot{f}_\alpha(t)|\nu_\alpha\rangle \\ &= \mathcal{H}_V|\nu(t)\rangle = \sum_\beta f_\beta(t)\mathcal{H}_V|\nu_\beta\rangle \\ &= \sum_\beta f_\beta(t)\sum_\alpha |\nu_\alpha\rangle\langle\nu_\alpha|\mathcal{H}_V|\nu_\beta\rangle \\ &= \sum_\alpha\left[\sum_\beta(\mathcal{H}_V)_{\alpha\beta}f_\beta(t)\right]|\nu_\alpha\rangle \ . \end{aligned} \tag{41}$$

Here, $(\mathcal{H}_V)_{\alpha\beta} \equiv \langle\nu_\alpha|\mathcal{H}_V|\nu_\beta\rangle$. Comparing the coefficients of $|\nu_\alpha\rangle$ at the beginning and end of Eq. (41), we clearly have

$$i\frac{\partial}{\partial t}\begin{bmatrix} f_e(t) \\ f_\mu(t) \end{bmatrix} = \mathcal{H}_V \begin{bmatrix} f_e(t) \\ f_\mu(t) \end{bmatrix} \quad , \tag{42}$$

where $\mathcal{H}_V$ is now the 2×2 matrix with elements $(\mathcal{H}_V)_{\alpha\beta}$. The Schrödinger equation (42) is completely analogous to the familiar one for a spin-1/2 particle. The roles of the two spin states are now being played by the two flavor states.

Let us call the two neutrino mass eigenstates out of which ν_e and ν_μ are made ν_1 and ν_2. To find the matrix $\mathcal{H}_V$, let us assume that our neutrino has a definite momentum p, so that its mass-eigenstate component ν_i has definite energy E_i given by Eq. (16). That is, $\mathcal{H}_V|\nu_i\rangle = E_i|\nu_i\rangle$, and the different mass eigenstates $|\nu_i\rangle$, like the eigenstates of any Hermitean Hamiltonian, are orthogonal to each other. Then, in view of Eq. (6), the elements $(\mathcal{H}_V)_{\alpha\beta}$ of the vacuum Hamiltonian are given by

$$(\mathcal{H}_V)_{\alpha\beta} \equiv \langle\nu_\alpha|\mathcal{H}_V|\nu_\beta\rangle = \langle\sum_i U^*_{\alpha i}\nu_i|\mathcal{H}_V|\sum_j U^*_{\beta j}\nu_j\rangle = \sum_i U_{\alpha i}U^*_{\beta i}E_i \; . \tag{43}$$

In this expression, the two-neutrino mixing matrix U may be taken from Eq. (32). However, we have seen that the complex phase factors in Eq. (32) have no effect on the two-neutrino oscillation probabilities, Eq. (33). Indeed, it is not hard to show that when there are only two neutrinos, complex phases in U have no effect whatsoever on neutrino oscillation. Thus, since oscillation is our only concern here, we may remove the complex phase factors from the U of Eq. (32). If, in addition, we relabel the mixing angle θ_V (short for θ_{Vacuum}), U becomes

$$U = \begin{bmatrix} \cos\theta_V & \sin\theta_V \\ -\sin\theta_V & \cos\theta_V \end{bmatrix} \quad . \tag{44}$$

Inserting in Eq. (43) the elements $U_{\alpha i}$ of this matrix and the energies E_i given by Eq. (16), we can obtain all the $(\mathcal{H}_V)_{\alpha\beta}$.

The matrix $\mathcal{H}_V$ can be put into a more symmetric and convenient form if we add to it a suitably chosen multiple $(\Delta E)I$ of the identity matrix I. Such an addition will not change the predictions of $\mathcal{H}_V$ for neutrino oscillation. To see why, we note first that the identity matrix is invariant under the unitary transformation that diagonalizes $\mathcal{H}_V$. Thus, adding $(\Delta E)I$ to $\mathcal{H}_V$ in the flavor basis, where its elements are $(\mathcal{H}_V)_{\alpha\beta}$, is equivalent to adding $(\Delta E)I$ to $\mathcal{H}_V$ in the mass eigenstate basis, where it is diagonal. Hence, if the eigenvalues of $\mathcal{H}_V$ are $E_i, i = 1, 2$, those of $\mathcal{H}_V + (\Delta E)I$ are $E_i + \Delta E, i = 1, 2$. That is, both eigenvalues are displaced by the same amount, ΔE. To see that such a

common shift of all eigenvalues does not affect neutrino oscillation, suppose a neutrino is born at time $t = 0$ with flavor α. That is, $|\nu(0)\rangle = |\nu_\alpha\rangle$. After a time t, this neutrino will have evolved into the state $|\nu(t)\rangle$ given, according to Schrödinger's equation, by

$$\begin{aligned} |\nu(t)\rangle &= e^{-i\mathcal{H}_V t}|\nu(0)\rangle \\ &= e^{-i\mathcal{H}_V t}\sum_i U^*_{\alpha i}|\nu_i\rangle = \sum_i U^*_{\alpha i}e^{-iE_i t}|\nu_i\rangle \quad . \end{aligned} \tag{45}$$

The amplitude $A(\nu_\alpha \to \nu_\beta)$ for this neutrino to have oscillated into a ν_β in the time t is then given by

$$A(\nu_\alpha \to \nu_\beta) = \langle \nu_\beta | \nu(t)\rangle = \sum_i U^*_{\alpha i} e^{-iE_i t} U_{\beta i} \quad . \tag{46}$$

Clearly, if we add $(\Delta E)I$ to $\mathcal{H}_V$ so that its eigenvalues E_i are replaced by $E_i + \Delta E$, then $A(\nu_\alpha \to \nu_\beta)$ is just multiplied by the overall phase factor $\exp[-i(\Delta E)t]$. Obviously, this phase factor has no effect on the oscillation probability $P(\nu_\alpha \to \nu_\beta) = |A(\nu_\alpha \to \nu_\beta)|^2$. Thus, the addition of $(\Delta E)I$ to $\mathcal{H}_V$ does not affect neutrino oscillation.

For our purposes, the most convenient choice of ΔE is $-[p+(M_1^2+M_2^2)/4p]$. Then, from Eq. (43), the new effective Hamiltonian $\mathcal{H}_V' \equiv \mathcal{H}_V + (\Delta E)I$ has the matrix elements

$$(\mathcal{H}_V')_{\alpha\beta} = \sum_i U_{\alpha i}U^*_{\beta i}E_i - [p + \frac{M_1^2 + M_2^2}{4p}]\delta_{\alpha\beta} \quad . \tag{47}$$

From Eqs. (44) and (16), this gives[7]

$$\mathcal{H}_V' = \frac{\delta M^2_{21}}{4E}\begin{bmatrix} -\cos 2\theta_V & \sin 2\theta_V \\ \sin 2\theta_V & \cos 2\theta_V \end{bmatrix} \quad . \tag{48}$$

Here, we have used the fact that $p \cong E$, the energy of the neutrino averaged over its two mass-eigenstate components. We leave to the reader the instructive exercise of verifying that, inserted into the Schrödinger Eq. (42), the $\mathcal{H}_V'$ of Eq. (48) does indeed lead to the usual two-neutrino oscillation probability, Eq. (33).

With the Hamiltonian that governs neutrino propagation through the vacuum in hand, let us now ask how neutrino propagation is modified by the presence of matter. Matter, of course, consists of electrons and nucleons. When passing through a sea of electrons and nucleons, a (non-sterile) neutrino can undergo the forward elastic scatterings depicted in Fig. (5). Coherent for-

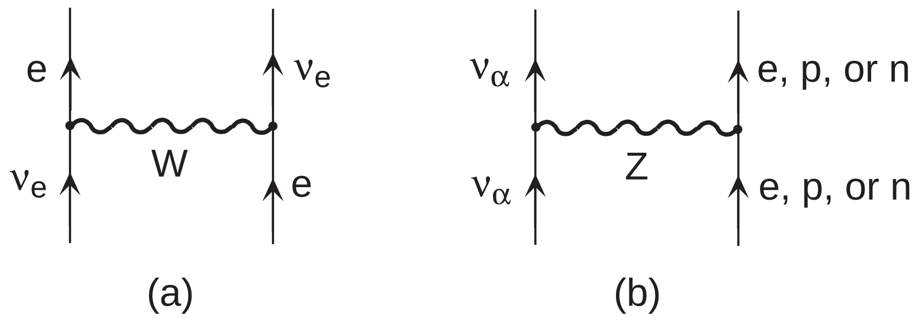

Figure 5: Forward elastic scattering of a neutrino from a particle of matter. (a) W-exchange-induced scattering from an electron, which is possible only for a ν_e. (b) Z-exchange-induced scattering from an electron, proton, or neutron. This is possible for $\nu_\alpha = \nu_e, \nu_\mu$, or ν_τ. According to the Standard Model, the amplitude for this Z exchange is the same, for any given target particle, for all three active neutrino flavors.

ward scatterings, via the pictured processes, from many particles in a material medium will give rise to an interaction potential energy of the neutrino in the medium. Since one of the reactions in Fig. (5) can occur only for electron neutrinos, this interaction potential energy will depend on whether the neutrino is a ν_e or not. The interaction potential energy for a neutrino of flavor α must be added to the matrix element $(\mathcal{H}'_V)_{\alpha\alpha}$ to obtain the Hamiltonian for propagation of a neutrino in matter.[8]

An important application of this physics is to the motion of solar neutrinos through solar material. The solar neutrinos are produced in the center of the sun by nuclear reactions such as $p + p \rightarrow {}^2H + e^+ + \nu_e$. The neutrinos produced by these reactions are all electron neutrinos. Let us suppose that the only neutrinos with which electron neutrinos mix appreciably are muon neutrinos, so that we have a two-neutrino system of the kind we have just been discussing. The solar neutrinos stream outward from the center of the sun in all directions, some of them eventually arriving at solar neutrino detectors here on earth. The passage of these solar neutrinos through solar material on their way out of the sun modifies their oscillation. Any neutrino which is still a ν_e, as it was at birth, can interact with solar electrons via the W exchange of Fig. (5a). This interaction leads to an interaction potential energy $V_W(\nu_e)$ of an electron neutrino in the sun. This $V_W(\nu_e)$ is obviously proportional to the Fermi coupling constant G_F, which governs the amplitude for the process in Fig. (5a). It is also proportional to the number of electrons per unit volume,

N_e, at the location of the neutrino, since N_e measures the number of electrons which can contribute coherently to the forward ν_e scattering. One can show[9] that in the Standard Model,

$$V_W(\nu_e) = \sqrt{2} G_F N_e \quad . \tag{49}$$

This energy must be added to $(\mathcal{H}'_V)_{ee}$ to obtain the Hamiltonian for propagation of a neutrino in the sun.

In principle, the interaction energy produced by the Z exchanges of Fig. (5b) must also be added to $\mathcal{H}'_V$. However, since these Z exchanges are both flavor diagonal and flavor independent, their contribution to the Hamiltonian is a multiple of the identity matrix. As we have already seen, a contribution of this character does not affect neutrino oscillation. Thus, we may safely ignore it.

In incorporating $V_W(\nu_e)$ into the Hamiltonian, it is convenient, in the interest of symmetry, to add as well the multiple $-\frac{1}{2}V_W(\nu_e)I$ of the identity matrix. Thus, with $\mathcal{H}'_V$ the Hamiltonian for propagation in vacuum given by Eq. (48), the Hamiltonian $\mathcal{H}_\odot$ for propagation in the sun is given by[8]

$$\begin{aligned} \mathcal{H}_\odot &= \mathcal{H}'_V + \frac{G_F}{\sqrt{2}} N_e \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \frac{\delta M^2_{21} D_\odot}{4E} \begin{bmatrix} -\cos 2\theta_\odot & \sin 2\theta_\odot \\ \sin 2\theta_\odot & \cos 2\theta_\odot \end{bmatrix} \quad . \end{aligned} \tag{50}$$

In this expression,

$$D_\odot = \sqrt{\sin^2 2\theta_\odot + (\cos 2\theta_\odot - x_\odot)^2} \, , \tag{51}$$

and

$$\sin^2 2\theta_\odot = \frac{\sin^2 2\theta_V}{\sin^2 2\theta_V + (\cos 2\theta_V - x_\odot)^2} \, , \tag{52}$$

where

$$x_\odot = \frac{2\sqrt{2} G_F N_e E}{\delta M^2_{21}} \quad . \tag{53}$$

The angle $\theta_\odot$ is the effective neutrino mixing angle in the sun when the electron density is N_e.

We note that $\mathcal{H}_\odot$, Eq. (50), has precisely the same form as $\mathcal{H}'_V$, Eq. (48). The only difference between these two Hamiltonians is that the parameters—the mixing angle and the effective neutrino (Mass)^2 splitting out in front of the matrix—have different values. Of course, the electron density N_e is not a constant, but depends on the distance r from the center of the sun. Thus,

the parameters $\theta_\odot$ and $(\delta M^2_{21})D_\odot$ are not constant either, unlike their counterparts, θ_V and δM^2_{21}, in $\mathcal{H}'_V$. However, let us imagine for a moment that N_e *is* a constant. Then $H_\odot$, like $\mathcal{H}'_V$, is independent of position, and must lead to the same oscillation probability, Eq. (33), as $\mathcal{H}'_V$ does, except for the substitutions $\theta \rightarrow \theta_\odot$ and $\delta M^2_{21} \rightarrow (\delta M^2_{21})D_\odot$. That is, in matter of constant electron density N_e, $H_\odot$ leads to the oscillation probability

$$\begin{aligned} P(\nu_e \rightarrow \nu_\mu) &= P(\nu_\mu \rightarrow \nu_e) \\ &= \sin^2 2\theta_\odot \sin^2[1.27\, \delta M^2_{21}(\mathrm{eV}^2) D_\odot \frac{L(\mathrm{km})}{E(\mathrm{GeV})}] \ . \end{aligned} \tag{54}$$

Now let N_e vary with r as it does in the real world. However, suppose that it varies slowly enough that the constant-N_e picture we have just painted applies at any given radius r, but with $N_e(r)$, hence $x_\odot(r)$, slowly decreasing as r increases. Suppose also that δM^2_{21} and E are such that $x_\odot(r=0) > \cos 2\theta_V$. Then, assuming that $\cos 2\theta_V > 0$, there must be a radius $r = r_c$ somewhere between $r = 0$ and the outer edge of the sun, where $N_e \rightarrow 0$, such that $x_\odot(r_c) = \cos 2\theta_V$. From Eq. (52), we see that at this special radius r_c, there is a kind of "resonance" with $\sin^2 2\theta_\odot = 1$, even if θ_V is tiny. That is, mixing can be maximal in the sun even if it is very small in vacuo. As a result, the oscillation probability, which is proportional to the mixing factor $\sin^2 2\theta_\odot$ as we see in Eq. (54), can be very large.

A nice picture of this enhanced probability for flavor transitions in matter can be gained by considering the neutrino energy eigenvalues and eigenvectors. If we neglect the inconsequential Z exchange contribution, and take $H_\odot$ from Eq. (50), then the true Hamiltonian for propagation in the sun is

$$\mathcal{H}_{\mathrm{True}} = \mathcal{H}_\odot + [p + \frac{M_1^2 + M_2^2}{4p} + \frac{1}{2} V_W(\nu_e)]\, I \ , \tag{55}$$

since the second term in this expression was subtracted from the true Hamiltonian to get $\mathcal{H}_\odot$. Now, $p + (M_1^2 + M_2^2)/4p \cong E$, the energy our neutrino would have in vacuum, averaged over its two mass-eigenstate components. Thus, from Eqs. (55) and (50), (48), (49), and (53), we have in the $\nu_e - \nu_\mu$ basis

$$\mathcal{H}_{\mathrm{True}}(r) = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} + \frac{\delta M^2_{21}}{4E} \begin{bmatrix} -\cos 2\theta_V + 2x_\odot(r) & \sin 2\theta_V \\ \sin 2\theta_V & \cos 2\theta_V \end{bmatrix} \ . \tag{56}$$

If we continue to assume that $N_e(r)$, and consequently $x_\odot(r)$, varies slowly, we may diagonalize this Hamiltonian for one r at a time to see how a solar neutrino will behave. We find from Eq. (56) that for a given r, the energy

eigenvalues $E_\pm(r)$ are given by

$$E_\pm(r) = E + \frac{\delta M_{21}^2}{4E}\left[x_\odot(r) \pm \sqrt{(x_\odot(r) - \cos 2\theta_V)^2 + \sin^2 2\theta_V}\right] \quad . \qquad (57)$$

To explore the implications of these energy levels, let us suppose that δM_{21}^2 is such that at $r = 0$, where $N_e(r)$ and hence $x_\odot(r)$ has its maximum value, $x_\odot \gtrsim 1$. From Eq. (53), the value of G_F, the value ($\sim 10^{26}$/cc) of $N_e(r = 0)$, and the typical energy E (~ 1 MeV) of a solar neutrino, we find that the required δM_{21}^2 is of order 10^{-5} eV^2. The dominant term in the energies $E_\pm(r)$ of Eq. (57) will be the first one, E (~ 1 MeV). However, very interesting physics will result from the second term, despite the fact that this term is only of order $\delta M_{21}^2/4E \sim (10^{-5}\text{eV}^2)/1\text{MeV} \sim 10^{-17}$ MeV!

The neutrino states which propagate in the sun without mixing significantly with each other are the eigenvectors of $\mathcal{H}_{\text{True}}(r)$. To study these eigenvectors, let us assume for simplicity that the vacuum mixing angle θ_V is small. Then it quickly follows that, except in the vicinity of the special radius r_c where $x_\odot(r_c) = \cos 2\theta_V$, one of the eigenvectors of $\mathcal{H}_{\text{True}}(r)$, Eq. (56), is essentially pure ν_e, while the other is essentially pure ν_μ. The evolution of a neutrino traveling outward through the sun is then as depicted in Fig. (6). The neutrino follows the trajectory indicated by the arrows. Produced by some nuclear process, the neutrino is born at small r as a ν_e. From Eq. (56), the eigenvector that is essentially ν_e at $r = 0$ is the one with the higher energy, $E_+(r)$. Thus, our neutrino begins its outward journey through the sun as the eigenvector belonging to the upper eigenvalue, E_+. Since the eigenvectors do not cross at any r and do not mix appreciably, the neutrino will remain this eigenvector. However, the flavor content of this eigenvector changes dramatically as the neutrino passes through the region near the radius $r = r_c$ where $x_\odot(r) - \cos 2\theta_V = 0$. For $r \ll r_c$, the eigenvector belonging to $E_+(r)$ is essentially a ν_e, as one may see from Eq. (56) if one neglects the small off-diagonal terms proportional to $\sin 2\theta_V$, and imposes the small-r condition $x_\odot > \cos 2\theta_V$. But for $r \gg r_c$, the eigenvector belonging to the higher energy level $E_+(r)$ is essentially a ν_μ, as one may also see from Eq. (56) if one neglects the off-diagonal terms and imposes the large-r condition $x_\odot < \cos 2\theta_V$. Thus, the eigenvector corresponding to the higher energy eigenvalue $E_+(r)$, along which the solar neutrino travels, starts out as a ν_e at the center of the sun, but ends up as a ν_μ at the outer edge of the sun. The solar neutrino, born a ν_e in the solar core, emerges from the rim of the sun as a ν_μ. Furthermore, it does this with high probability even if the vacuum mixing angle θ_V is very small, so that oscillation in vacuum would not have much of an effect. This very efficient conversion of solar electron neutrinos into neutrinos of another flavor as a result of interaction with matter

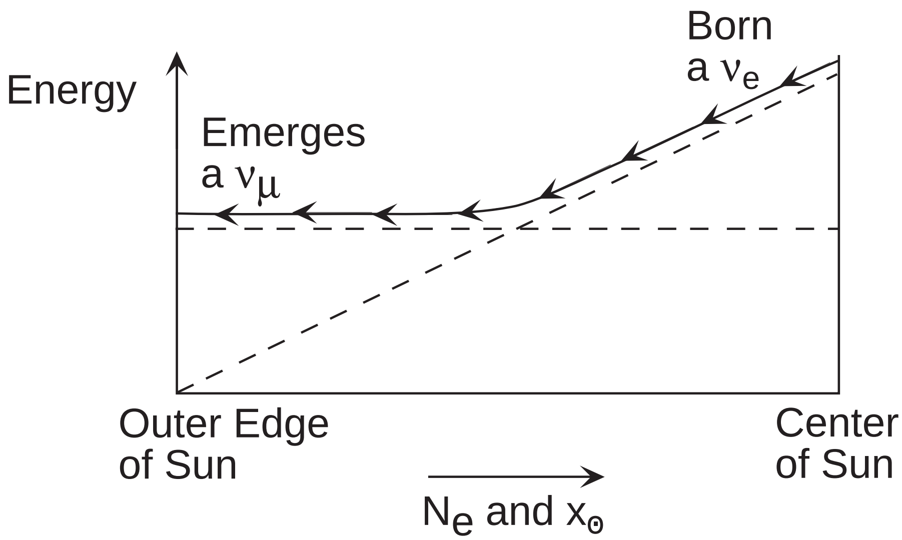

Figure 6: Propagation of a neutrino in the sun. The horizontal axis is linear in N_e and $x_\odot$. The solid line shows the upper eigenvalue, $E_+(r)$, for small θ_V. The dashed lines show the two eigenvalues, $E_\pm(r)$, when there is no vacuum mixing ($\theta_V = 0$). A neutrino born as a ν_e follows the path indicated by the arrows.

is known as the Mikheyev-Smirnov-Wolfenstein (MSW) effect.[8,10]

We turn now from solar neutrinos to atmospheric neutrinos, whose propgation through the earth entails a second important application of the physics of neutrinos traveling through matter. As already mentioned, an atmospheric neutrino can be produced in the atmosphere on one side of the earth, and then journey through the whole earth to be detected in a detector on the other side. While traveling through the earth, this neutrino will undergo interactions that can significantly modify its oscillation pattern.

There is very strong evidence[11] that the atmospheric neutrinos born as muon neutrinos oscillate into neutrinos ν_x of another flavor. It is known that ν_x is not a ν_e. It could be a ν_τ, or a sterile neutrino ν_s, or sometimes one of these and sometimes the other. One way to find out what is becoming of the oscillating muon neutrinos is to see whether their oscillation is afffected by their passage through earth-matter. The oscillation $\nu_\mu \to \nu_s$ will be affected, but $\nu_\mu \to \nu_\tau$ will not be.

To see why this is so, let us first assume that the oscillation is $\nu_\mu \to \nu_\tau$.

Then the Hamiltonian $\mathcal{H}_E$ (short for $\mathcal{H}_{\text{Earth}}$) that describes neutrino propagation in the earth is a 2×2 matrix in $\nu_\mu - \nu_\tau$ space. Now, either a ν_μ or a ν_τ will interact with earth-matter via the Z exchange of Fig. (5b). This interaction will give rise to an interaction potential energy of the neutrino in the earth. However, according to the S(tandard) M(odel), the Z-exchange amplitude is the same for a ν_τ as it is for a ν_μ. Thus, in the case of $\nu_\mu \to \nu_\tau$ oscillation, the contribution of neutrino-matter interaction to $\mathcal{H}_E$ is a multiple of the identity matrix. As we have alrady seen, such a contribution has no effect on oscillation.

Now, suppose the oscillation is not $\nu_\mu \to \nu_\tau$ but $\nu_\mu \to \nu_s$. Then $\mathcal{H}_E$ is a 2×2 matrix in $\nu_\mu - \nu_s$ space. A ν_μ will interact with earth-matter, as we have discussed, but a ν_s, of course, will not. Thus, the contribution of neutrino-matter interaction to $\mathcal{H}_E$ is now of the form

$$\begin{array}{cc} & \begin{array}{cc} \nu_\mu & \nu_s \end{array} \\ \begin{array}{c} \nu_\mu \\ \nu_s \end{array} & \begin{bmatrix} V_Z(\nu_\mu) & 0 \\ 0 & 0 \end{bmatrix} \end{array} \quad , \tag{58}$$

where $V_Z(\nu_\mu)$ is the interaction potential energy of muon neutrinos produced by the Z exchange of Fig. (5b). Since the matrix (58) is not a multiple of the identity, neutrino-matter interaction does affect $\nu_\mu \to \nu_s$ oscillation.

According to the SM, the forward Z-exchange amplitudes for a target e and a target p are equal and opposite. Thus, assuming that the earth is electrically neutral so that it contains an equal number of electrons and protons per unit volume, the e and p contributions to $V_Z(\nu_\mu)$ cancel. Then $V_Z(\nu_\mu)$ is proportional to the neutron number density, N_n. Taking the proportionality constant from the SM, we have

$$V_Z(\nu_\mu) = -\frac{G_F}{\sqrt{2}} N_n \quad . \tag{59}$$

To obtain the Hamiltonian $\mathcal{H}_E$ for $\nu_\mu \to \nu_s$ oscillation in the earth, we add to the vacuum Hamiltonian $\mathcal{H}'_V$ of Eq. (48) the contribution (58) from matter interactions, using Eq. (59) for $V_Z(\nu_\mu)$. *Of course, it must be understood that $\mathcal{H}'_V$ is now to be taken as a matrix in $\nu_\mu - \nu_s$ space, and that the vacuum (Mass)^2 splitting δM^2_{21} and mixing angle θ_V in $\mathcal{H}'_V$ are now different parameters than they were when we obtained from $\mathcal{H}'_V$ the Hamiltonian $\mathcal{H}_\odot$ for neutrino propagation in the sun. The quantities δM^2_{21} and θ_V are now new parameters appropriate to the vacuum oscillation of atmospheric, rather than solar, neutrinos.* To obtain a more symmetrical and convenient $\mathcal{H}_E$ from $\mathcal{H}'_V$, we also add $-\frac{1}{2} V_Z(\nu_\mu) I$, a multiple of the identity which will not affect the

implications of $\mathcal{H}_E$ for oscillation. The result is

$$\mathcal{H}_E = \frac{(\delta M_{21}^2)D_E}{4E}\begin{bmatrix} -\cos 2\theta_E & \sin 2\theta_E \\ \sin 2\theta_E & \cos 2\theta_E \end{bmatrix} \quad . \tag{60}$$

Here,

$$D_E = \sqrt{\sin^2 2\theta_V + (\cos 2\theta_V - x_E)^2} \quad , \tag{61}$$

and

$$\sin^2 2\theta_E = \frac{\sin^2 2\theta_V}{\sin^2 2\theta_V + (\cos 2\theta_V - x_E)^2} \; , \tag{62}$$

where

$$x_E = -\frac{\sqrt{2}G_F N_n E}{\delta M_{21}^2} \quad . \tag{63}$$

As a rough approximation, we may take the neutron density N_n to be constant throughout the earth. Then, x_E is also a constant, and the Hamiltonian $\mathcal{H}_E$ of Eq. (60) is identical to the vacuum Hamiltonian $\mathcal{H}_V'$ of Eq. (48), except that the constant δM_{21}^2 is replaced by the constant $(\delta M_{21}^2)D_E$, and the constant θ_V by the constant θ_E. Thus, from the fact that $\mathcal{H}_V'$ leads to the vacuum oscillation probability of Eq. (33) (with $\nu_e \rightarrow \nu_\mu$ replaced by $\nu_\mu \rightarrow \nu_s$ for the present application), we immediately conclude that the $\mathcal{H}_E$ of Eq. (60) leads to the oscillation probability

$$P(\nu_\mu \rightarrow \nu_s) = \sin^2 2\theta_E \sin^2[1.27\, \delta M_{21}^2(\mathrm{eV}^2) D_E \frac{L(\mathrm{km})}{E(\mathrm{GeV})}] \quad . \tag{64}$$

As we see from Eq. (63), x_E, which is a measure of the influence of matter effects on atmospheric neutrinos, grows with energy E. In a moment we will see that for $E \sim 1$ GeV, matter effects are negligible. Fits to data[12] on atmospheric neutrinos with roughly this energy have led to the conclusion that atmospheric neutrino oscillation involves a neutrino (Mass)^2 splitting $\delta M_{\mathrm{Atmos}}^2$ given by

$$\delta M_{\mathrm{Atmos}}^2 \sim 3 \times 10^{-3} \mathrm{eV}^2 \quad , \tag{65}$$

and a neutrino mixing angle θ_{Atmos} given by

$$\sin^2 2\theta_{\mathrm{Atmos}} \sim 1 \quad . \tag{66}$$

That is, the mixing when matter effects are negligible is very large, and perhaps maximal. The quantities $\delta M_{\mathrm{Atmos}}^2$ and θ_{Atmos} are to be taken, respectively, for δM_{21}^2 and θ_V in Eqs. (60) - (64) to find the implications of those equations for $\nu_\mu \rightarrow \nu_s$ within the earth.

Since atmospheric neutrino oscillation involves maximal mixing when matter effects are negligible, the matter effects cannot possibly enhance the oscillation, but can only suppress it. From Eq. (62), we see that if, as observed, $\sin^2 2\theta_V \simeq 1$, then matter effects will lead to a smaller effective mixing $\sin^2 2\theta_E$ in the earth, given by

$$\sin^2 2\theta_E = \frac{1}{1+x_E^2} \quad . \tag{67}$$

As is clear in Eq. (64), this will result in a smaller oscillation probability $P(\nu_\mu \to \nu_s)$ than one would have in vacuum, where $\sin^2 2\theta_E$ is replaced by $\sin^2 2\theta_V$ (~ 1).

Since x_E grows with energy, the degree to which matter effects suppress $\nu_\mu \to \nu_s$ grows as well. From Eq. (63) for x_E, the known values of G_F and N_n, and the value (65) required for δM^2_{21} by the data, we find that $x_E \ll 1$ when $E \sim 1$ GeV. Thus, matter effects are indeed negligible at this energy, so it is legitimate to determine[12] the vacuum parameters $\delta M^2_{\mathrm{Atmos}}$ and $\sin^2 2\theta_{\mathrm{Atmos}}$ by analyzing the ~ 1 GeV data neglecting matter effects. However, at sufficiently large E, the matter-induced suppression of $\nu_\mu \to \nu_s$ will obviously be significant. From Eq. (63), we find that $\sin^2 2\theta_E$ is below 1/2 when $E \gtrsim 20$ GeV. The consequent suppression of oscillation at these energies has been looked for, and is not seen.[13] This absence of suppression is a powerful part of the evidence that the neutrinos into which the atmospheric muon neutrinos oscillate are not sterile neutrinos, or at least not solely sterile neutrinos.[13]

4 Conclusion

Evidence has been reported that the solar neutrinos, the atmospheric neutrinos, the accelerator-generated neutrinos studied by the Liquid Scintillator Neutrino Detector (LSND) experiment at Los Alamos, and the accelerator-generated neutrinos studied by the K2K experiment in Japan, actually do oscillate. Some of this evidence is very strong. The neutrino oscillation experiments, present and future, are discussed in this Volume by John Wilkerson.[14]

In these lectures, we have tried to explain the basic physics that underlies neutrino oscillation, and that is invoked to understand the oscillation experiments. As we have seen, neutrino oscillation implies neutrino mass and mixing. Thus, given the compelling evidence that at least some neutrinos do oscillate, we now know that neutrinos almost certainly have nonzero masses and mix. This knowledge rasises a number of questions about the neutrinos:

- How many neutrino flavors, including both interacting and possible sterile flavors, are there? Equivalently, how many neutrino mass eigenstates are there?

- What are the masses, M_i, of the mass eigenstates ν_i?
- Is the antiparticle $\overline{\nu_i}$ of a given mass eigenstate ν_i the same particle as ν_i, or a different particle?
- What are the sizes and phases of the elements $U_{\alpha i}$ of the leptonic mixing matrix? Equivalently, what are the mixing angles and complex phase factors in terms of which U may be described? Do complex phase factors in U lead to CP violation in neutrino behavior?
- What are the electromagnetic properties of neutrinos? In particular, what are their dipole moments?
- What are the lifetimes of the neutrinos? Into what do they decay?
- What is the physics that gives rise to the masses, the mixings, and the other properties of the neutrinos?

Seeking the answers to these and other questions about the neutrinos will be an exciting adventure for years to come.

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